FORMATION OF AN ICE LAYER OF GIVEN THICKNESS BY ACCUMULATION OF A LIQUID ON A COOLED-CYLINDRICAL SURFACE

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UDC 536.421.4

An approximate solution is given to the problem of accumulation of a liquid on a cylindrical surface, taking into account thermal resistivity. Particular cases are considered.

In the accumulation of a liquid on a highly-cooled-cylindrical surface, the problem arises of determining the formation time for a frozen layer of a given thickness. One meets with such problems, for example, in the food industry in the operation of generators of squamate ice.

In this case, allowance for the influence of the thermal resistivity of the material of the cylindrical wall of the cooler and its constructional dimensions turn out to be very significant.

Mathematically, the problem amounts to the solution of the equations

$$c_1\gamma_1 \frac{\partial t_1}{\partial \tau} = \lambda_1 \left(\frac{\partial^2 t_1}{\partial r^2} + \frac{1}{r} \frac{\partial t_1}{\partial r} \right), \quad r_0 \leqslant r \leqslant r_0 + R, \tag{1}$$

$$c\gamma \frac{\partial t}{\partial \tau} = \lambda \left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \frac{\partial t}{\partial r} \right), \quad r_0 + R \leqslant r \leqslant r_0 + R + \xi \tag{2}$$

under the conditions of conjugation

$$t_1 |_{r=r_0+R} = t |_{r=r_0+R} = T(\xi), \tag{3}$$

$$\lambda_1 \frac{\partial t_1}{\partial r} \bigg|_{r=r_0+R} = \lambda \frac{\partial t}{\partial r} \bigg|_{r=r_0+R}, \tag{4}$$

with the boundary conditions

$$t_1 |_{r=r_0} = t_0, \tag{5}$$

$$\lambda \frac{\partial t}{\partial r} \bigg|_{r=r_0+R+\xi} = \alpha (t_c - t_{cr}) + \gamma\rho \frac{d\xi}{d\tau}, \tag{6}$$

$$t |_{r=r_0+R+\xi} = t_{cr} \tag{7}$$

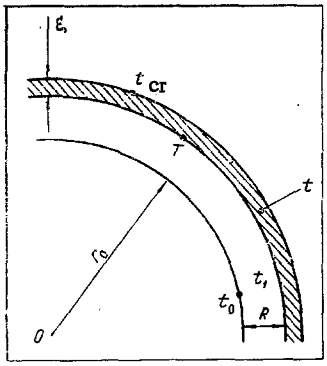

Fig. 1. Formation of an ice layer on a cylindrical surface.

and with given initial conditions

Thus, we consider the problem with a first-order boundary condition on the inner surface and a third-order boundary condition on the outer surface.

It is impossible to find an exact analytical solution of the problem. Many varied approximate methods exist for the solution of such moving boundary problems. They are partially enumerated in the interesting article by A. M. Makarov [1]. In the present work, a practical method of solution is offered which is based on the replacement of the true temperature curves by their simplest analogs. Such approximation is accepted

Leningrad Technological Institute of the Refrigeration Industry. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 27, No. 4, pp. 667-671, October, 1974. Original article submitted March 29, 1974.

in many works. In particular, L. S. Leibenzon [2] used this method in his thorough investigations.

We will assume the temperature in both intervals (r_0, $r_0 + R$) and ($r_0 + R$, $r_0 + R + \xi$) to be in accordance with the laws which are obtained in a steady-state distribution in a cylindrical wall, namely, in the form

$$t_1 = C_1 \ln r + C_2 \text{ and } t = C_3 \ln r + C_4.$$

Using the conditions in (5), (3) and (7), we find the value of the constants C_i ($i = 1, 2, 3, 4$) and then

$$t_1 = \frac{T - t_0}{\ln \frac{r_0 + R}{r_0}} [\ln r - \ln r_0] + t_0, \tag{8}$$

$$t = \frac{t_{cr} - T}{\ln \frac{r_0 + R + \xi}{r_0 + R}} [\ln r - \ln (r_0 + R)] + T. \tag{9}$$

We find the value T from the conjugation condition (4) after differentiating (8) and (9)

$$T = \frac{\lambda_1 t_0 \ln \frac{r_0 + R + \xi}{r_0 + R} + \lambda t_{cr} \ln \frac{r_0 + R}{r_0}}{\lambda_1 \ln \frac{r_0 + R + \xi}{r_0 + R} + \lambda \ln \frac{r_0 + R}{r_0}}. \tag{10}$$

This equation is useful in itself, since it allows approximate estimation of the temperature T at the phase interface as a function of the thickness of the ice layer ξ which has been formed.

Substituting (9) into (6), with allowance for (10), we obtain, after the obvious transformations, an ordinary differential equation with separable variables (with $t_{cr} = 0$):

$$\rho\gamma \frac{d\xi}{d\tau} = -\frac{\lambda\lambda_1 t_0 + \alpha t_c (r_0 + R + \xi) \left[\lambda_1 \ln \frac{r_0 + R + \xi}{r_0 + R} + \lambda \ln \frac{r_0 + R}{r_0} \right]}{(r_0 + R + \xi) \left[\lambda_1 \ln \frac{r_0 + R + \xi}{r_0 + R} + \lambda \ln \frac{r_0 + R}{r_0} \right]},$$

whence

$$\tau = -\frac{\rho\gamma}{\alpha t_c} \int\limits_0^{\xi} \frac{\left(1 + \frac{\xi}{r_0 + R}\right) \left[\ln \left(1 + \frac{\xi}{r_0 + R}\right) + B \right] d\xi}{\left(1 + \frac{\xi}{r_0 + R}\right) \left[\ln \left(1 + \frac{\xi}{r_0 + R}\right) + B \right] - A}, \tag{11}$$

where

$$A = -\frac{\lambda t_0}{\alpha t_c (r_0 + R)}, \quad B = \frac{\lambda}{\lambda_1} \ln \left(\frac{r_0 + R}{r_0}\right). \tag{12}$$

Since the integral in (11) cannot be expressed in terms of elementary functions, then, using the smallness of the value

$$\frac{\xi}{r_0 + R} = \eta \tag{13}$$

and changing the integrand to the form

$$\frac{\ln(1 + \eta) + B}{\ln(1 + \eta) + B - \frac{A}{1 + \eta}},$$

we expand the numerator and denominator of the last expression into power series. Designating the quotient from division of the series in the form of a series with undetermined coefficients

$$a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + \ldots, \tag{14}$$

we obtain the identity

$$B+\eta-\frac{\eta^2}{2}+\frac{\eta^3}{3}-\cdots=\left[(B-A)+(1+A)\eta-\left(\frac{1}{2}+A\right)\eta^2+\cdots\right](a_0+a_1\eta+a_2\eta^2+a_3\eta^3+\cdots),$$

from which, comparing the coefficients of identical powers of η, we find the interesting numbers

$$a_0=\frac{B}{B-A},\quad a_1=-\frac{A(1+B)}{(B-A)^2},\quad a_2=\frac{A(A+3B+2B^2+2)}{2(B-A)^3}.$$

Integrating the series (14), we will find from Eq. (11)

$$\tau=-\frac{\gamma\rho}{\alpha t_c}\left[\frac{B\xi}{B-A}-\frac{A(1+B)\xi^2}{2(B-A)^2(r_0+R)}+\frac{A(A+3B+2B^2+2)\xi^3}{6(B-A)^3(r_0+R)^2}+\cdots\right]. \tag{15}$$

For practical calculations, as a result of the smallness of ξ it is adequate to limit oneself to two terms of the series (15); we then obtain the final standard working equation

$$\tau=-\frac{\gamma\rho}{\alpha t_c}\left[\frac{B}{B-A}\xi-\frac{A(1+B)}{2(B-A)^2(r_0+R)}\xi^2\right]. \tag{16}$$

One can generalize the last equation somewhat to the case of a third-order boundary condition on the inner surface of a cylindrical drum (i.e., with $r = r_0$). However, in the majority of cases the heat-transfer coefficient for the medium occupying the interior of the drum (the coolant) is so large that it is hardly advisable to complicate an approximate equation (16) whose convenience for engineering calculations lies in its simplicity.

From Eq. (16) it is possible to obtain a number of particular cases:

1. If the thickness of the drum wall is not taken into account, it follows that $R = 0$; then $B = 0$ and

$$\tau=-\frac{\gamma\rho\xi^2}{2\lambda t_0}. \tag{17}$$

2. If we let r_0 approach infinity, then, by passage to the limit, we will obtain appropriate equations for a plate of thickness R instead of equations for a cylindrical wall. Here

$$\frac{B-A}{B}\to 1+\frac{\lambda_1 t_0}{\alpha t_c R},$$

$$\frac{A(1+B)}{2(B-A)^2(r_0+R)}\to-\frac{\lambda_1^2\alpha t_0 t_c}{2\lambda(\lambda_1 t_0+\alpha t_c R)^2}$$

and instead of (16) we will have

$$\tau=\gamma\rho\left[-\frac{R}{\lambda_1 t_0+\alpha t_c R}\xi-\frac{t_0\lambda_1^2}{2\lambda(\lambda_1 t_0+\alpha t_c R)^2}\xi^2\right], \tag{18}$$

which is readily found also from the equation

$$\tau=-\frac{\gamma\rho}{\alpha t_c}\left[\xi-\frac{\lambda t_0}{\alpha t_c}\ln\left(1+\frac{\alpha t_c\lambda_1\xi}{\lambda(\alpha t_c R+\lambda_1 t_0)}\right)\right], \tag{19}$$

obtained in [3] if we expand the logarithm into a series and limit ourselves to only its first two terms. Let us note, incidentally, that it is preferable to use Eq. (18) rather than (19), since we thus avoid the subtraction of near numbers.

3. If we disregard the thickness of the plate, then, instead of (18), we obtain an equation coincident with (17):

$$\tau=-\frac{\gamma\rho}{2\lambda t_0}\xi^2. \tag{20}$$

Thus, with third-order accuracy of smallness in regard to the thickness of the frozen layer ξ, if the thickness of the metal wall R is not taken into consideration, the equations for a flat wall and a cylindrical one coincide. If thermal resistivity is taken into account, the formation time for an ice layer on a cylindrical wall will differ from that of an ice layer on a flat wall; moreover, the smaller the radius of the cylinder, the greater this difference will be.

The numerical values obtained according to the approximate equations (16) and (18) agree well with experimental results; this was verified in a series of experiments. The influence of wall thickness and the thermal conductivity of the wall material on the formation time of the ice layer τ and on the value ξ proved to be especially significant. It is possible to assess this according to the equations derived.

NOTATION

τ, is the time; t, is the ice temperature; t_1, is the wall temperature; T is the phase-interface temperature; t_c, is the temperature of the liquid undergoing freezing; t_{cr}, is the cryoscopic temperature; $\lambda(\lambda_1)$, is the thermal conductivity coefficient of the ice (wall); $c(c_1)$, is the specific heat of the ice (wall); $\gamma(\gamma_1)$, is the ice (wall) density; α, is the heat transfer coefficient; ρ, is the specific heat of ice formation.

LITERATURE CITED

1. A. M. Makarov et al., Inzh.-Fiz. Zh., 21, No. 3 (1971).
2. L. S. Leibenzon, Izv. Akad. Nauk SSSR, Ser. Geograf. i Geofizich., No. 6 (1939).
3. V. B. Rzhevskaya, L. A. Stepanova, and N. V. Fomin, Kholod. Tekh., No. 5 (1973).